Quantum Estimation in Open Systems: Dissipative Quantum Cramér-Rao Bound

S. Alipour and A. T. Rezakhani
Department of Physics, Sharif University of Technology, Tehran, Iran

Estimation of parameters is a pivotal task throughout science and technology. In quantum mechanics, quantum Cramér-Rao bound provides a fundamental limit of precision allowed to achieve under quantum theory. For closed quantum systems, it has been shown how the estimation precision depends on the underlying dynamics. Here, we propose a general, alternative formulation for an estimation scenario in open quantum systems, aiming to relate the precision more directly to properties of the underlying dynamics. Specifically, we derive a Cramér-Rao bound for a fairly large class of open system dynamics, which is governed by a Lindbladian master equation. We illustrate the utility of this scenario through an example.

PACS numbers: 03.65.Ta, 03.67.Lx, 06.20.Dk

Introduction.—Metrology and parameter estimation lie in the heart of science, and are prevalent in any aspect of technology. The basic task of identification or estimation of a set of unknown parameters essentially requires an inference from a pool of observed data about the parameters or the system to which they are attributed. As errors and imperfections are unavoidable in practice, increasing accuracy of the underlying tasks of data acquisition and inference—hence improving the quality of estimation—is an important goal of metrology [1]. Improving quality of measurement instruments and removing potential sources of systematic errors aside, statistics provides useful suggestions for enhancing metrology, such as increasing data size and repeated measurements on an ensemble of N 'probe' systems. Additionally (and more interestingly), the underlying physics of the system of interest may also dictate some restrictions or bounds on the ultimate achievable accuracy (usually described through a 'Cramér-Rao inequality' [2]), or even may offer new possibilities to exploit.

In quantum mechanics, measurements act differently than in classical systems. In addition, interactions with an environment or other systems as well as (quantum) correlations can each affect observed data [3], hence introduce new playing factors in estimation theory. For example, it has been shown that entanglement in a probe ensemble can be exploited to the advantage of a quantum estimation task [4], so that it enables the estimation error of $O(1/N)$ (the "Heisenberg limit"), in contrast to the classical statistical limit of $O(1/\sqrt{N})$ (the "shot-noise limit"). Alternatively, enabling k-body ($k \geqslant 2$) interactions among quantum probe systems has been shown to allow an error of $O(1/\sqrt{N^k})$ [5]; or, it has been argued that application of a suitable entangling operator may even offer an error as small as $O(2^{-N})$ [6] (beyond the Heisenberg limit). Moreover, nonclassicality has been examined as a potential resource for increasing the metrology resolution in quantum optics [7] (for a general framework of resource analysis, see, e.g., Ref. [8]). It thus seems natural to expect that some properties of quantum systems can be employed as a useful "resource" for quantum metrology.

In open quantum systems, due to interaction with an environment, the underlying dynamics becomes 'noisy.' As a result, formulation and analysis of quantum estimation also becomes more involved [9, 10]. In general, dynamics of an open system can be described as $\varrho_S(t) = \mathrm{Tr}_E[U_{SE}(t,t_0)\varrho_{SE}(t_0)U^{\dagger}_{SE}(t,t_0)]$, where ϱ_{SE} is the state of the systems and environment (SE), and $U_{SE}(t,t_0)$ is the corresponding unitary evolution [11, 12]. Thereby one can argue that in general there may exist a flow of information between the system and the environment [13]. Under some conditions, this dynamics can feature quantum Markovian or non-Markovian properties [14]. The former case typically appears when the environment has a small decoherence time during which correlations typically disappear, whereas in the latter correlations (both classical and/or quantum [16]) with the environment would form and persist. Such correlations are in practice inevitable, which necessitates investigation of 'noisy' quantum metrology [9, 17, 18]. Additionally, such correlations may also offer new resources for enhancing estimation tasks. A general analysis of the latter possibility is still lacking (see Ref. [19] for an example along this line).

In this article, we first lay out a fairly general formalism for open quantum system metrology. This (re)formulation of the problem (cf. Ref. [9]) has this advantage that here precision of estimation is more directly related to the underlying dynamics; besides, it is in some sense analogous to the closed system formulation. In particular, we derive a quantum Cramér-Rao bound (QCRB) for open system dynamics generated through dynamical map with semigroup property. We next illustrate this setting through an example. This example shows that how induced correlations of probe quantum systems through a common environment may offer relatively higher precision for an estimation task in a sense akin to what manybody interactions enable.

Open system dynamics.—Under some specific conditions, the dynamical equation describing the state of an open system ϱ_S [defined on a Hilbert space $\mathcal{H}_S$, i.e., $\varrho_S \in \mathcal{S}(\mathcal{H}_S)$, where $\mathcal{S}(\mathcal{H}_S)$ is the space of all linear operators acting on $\mathcal{H}$] reduces to $\partial_\tau \varrho_S(\tau) = \mathcal{L}_\tau[\varrho_S(\tau)]$, or equivalently $\varrho_S(\tau) = \mathbf{T}e^{\int_{\tau_0}^{\tau}\mathcal{L}_{\tau'}\mathrm{d}\tau'}[\varrho_S(\tau_0)]$, in which $\mathbf{T}$ denotes time-ordering, and $\mathcal{L}_\tau[\circ] = -i[H_S(\tau),\circ] + \sum_k \eta_k(\tau)(A_k(\tau)\circ A_k^{\dagger}(\tau) - (1/2)\{A_k^{\dagger}(\tau)A_k(\tau),\circ\})$ (for some set of operators $\{A_k(\tau)\}$) is the (Lindbladian) generator of the dynamical map, with $H_S(\tau)$ being the system Hamiltonian up to a Lamb shift term (we omit subscript S henceforth). We have also

assumed $\hbar \equiv 1$. In (time-dependent) Markovian evolutions, we have $\eta_k(\tau) \geqslant 0\ \forall k, \tau$; while if some η_k becomes negative for some intervals, the associated dynamics would be non-Markovian [11, 12, 14, 15].

Let us assume that a set of unknown parameters $\mathbf{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_\ell)$ are to be estimated in a quantum system subject to interaction with an environment. For simplicity, here we only consider the single parameter case (x). In the closed-system scenario, this parameter is usually assumed to enter into the dynamics as a linear coupling in the Hamiltonian $H(\mathrm{x}) = \mathrm{x}\mathsf{H}$ acting on some known initial state. In the open-system scenario, similarly the devised dynamics would in general depend on x, whence the dynamics of the system is described by $\partial_\tau \varrho(\mathrm{x}, t) = \mathcal{L}_\tau(\mathrm{x})[\varrho(\mathrm{x}, \tau)]$. For our analysis in this article, as we argue later, vectorization is an appropriate language. Vectorization of this equation yields $\partial_\tau |\varrho\rangle\rangle = \widetilde{\mathcal{L}}_\tau(\mathrm{x})|\varrho\rangle\rangle$, where $\widetilde{\mathcal{L}}_\tau$ is the matrix representation of $\mathcal{L}_\tau$ [20]. Defining the normalized pure state $\widetilde{\varrho} \equiv |\varrho\rangle\rangle\langle\langle\varrho|/\mathrm{Tr}[\varrho^2] \in \mathcal{S}(\mathcal{H}^{\otimes 2})$ and assuming that $\mathcal{L}_\tau$ does not depend on time, we have

$$\widetilde{\varrho}(\mathrm{x}, \tau) = e^{\tau \widetilde{\mathcal{L}}(\mathrm{x})} \widetilde{\varrho}(0) e^{\tau \widetilde{\mathcal{L}}^\dagger(\mathrm{x})} / \mathrm{Tr}[e^{\tau \widetilde{\mathcal{L}}(\mathrm{x})} \widetilde{\varrho}(0) e^{\tau \widetilde{\mathcal{L}}^\dagger(\mathrm{x})}]. \quad (1)$$

The initial preparation $\widetilde{\varrho}(0)$ may itself depend on x, but here we do not assume such generality.

QCRB for open system estimation.—Given a data set $D \equiv \{\gamma_i\}$ constituted from some measurement outcomes γ_i over N (identical) probe systems, an estimator $\mathrm{x}_{\mathrm{est}}(D)$ is chosen for the true value x. By repeating this scenario M times and averaging, the precision of the estimation of x, evaluated by $\delta\mathrm{x} = \sqrt{\mathrm{Var}(\mathrm{x})}$, is then fundamentally limited by the QCRB

$$\delta \mathrm{x} \geqslant 1/\sqrt{M}\sqrt{\mathcal{F}^{(\mathrm{Q})}(\mathrm{x}; N)}. \quad (2)$$

Here, $\mathrm{Var}(\mathrm{x})$ is the variance of any unbiased estimator $\mathrm{x}_{\mathrm{est}}(D)$ (for which, by definition, $\langle \mathrm{x}_{\mathrm{est}} \rangle = \mathrm{x}$, with $\langle \circ \rangle$ denoting the average with respect to the underlying quantum probability distribution), and $\mathcal{F}^{(\mathrm{Q})}(\mathrm{x}; N)$ is the so-called "quantum Fisher information" (QFI) [17, 21, 22]. By assuming the state of each N-probe set to be $\varrho^{(N)}(\mathrm{x}, \tau)$ (hereafter we omit superscript N for brevity) and defining the corresponding symmetric logarithmic derivative L_ϱ through $\partial_\mathrm{x} \varrho = (L_\varrho \varrho + \varrho L_\varrho)/2$, the QFI is defined as $\mathcal{F}^{(\mathrm{Q})}(\mathrm{x}, \tau; N) = \mathrm{Tr}[\varrho(\mathrm{x}, \tau) L^2_{\varrho(\mathrm{x}, \tau)}]$.

We remind that in *closed* systems, noting $\varrho(\mathrm{x}, \tau) = e^{-i\tau H(\mathrm{x})} \varrho(0) e^{i\tau H(\mathrm{x})}$, the spectral decomposition $\varrho = \sum r_i |r_i\rangle\langle r_i|$, and $L_\varrho = 2\sum_{ij} \langle r_i|\partial_\mathrm{x} \varrho|r_j\rangle/(r_i + r_j)|r_i\rangle\langle r_j|$ provides a relatively straightforward relation between $\mathcal{F}^{(\mathrm{Q})}$ and the interaction H. In particular, when $H(\mathrm{x}) \equiv \mathrm{x}\mathsf{H}$ and ϱ is pure, one obtains

$$\mathcal{F}^{(\mathrm{Q})} = 4\tau^2 \mathrm{Cov}_\varrho(\mathsf{H}, \mathsf{H}) \quad (3)$$

(with equality replaced with $\leqslant$ for mixed ϱ), where $\mathrm{Cov}_\varrho(X, Y) \equiv \langle XY \rangle_\varrho - \langle X \rangle_\varrho \langle Y \rangle_\varrho$ is the covariance function of a pair of observables X and Y, which here is the very quantum standard deviation $\Delta^2\mathsf{H}$ (with $\langle \circ \rangle_\varrho \equiv \mathrm{Tr}[\varrho \circ]$). The resulting relation

$$\delta \mathrm{x} \geqslant 1/\big(2\tau\sqrt{M}\sqrt{\mathrm{Cov}_\varrho(\mathsf{H}, \mathsf{H})}\big) = 1/(2\tau\sqrt{M}\Delta\mathsf{H}), \quad (4)$$

where $\Delta\mathsf{H} \equiv \sqrt{\Delta^2\mathsf{H}}$, is more in the spirit of an uncertainty-like relation [21], and shows explicitly how the precision is dictated by the interaction. In *open*-system cases, however, deriving similar, straightforward relations is hardly possible (except for some special cases [13]) since, e.g., calculating $L_{\varrho(\mathrm{x}, \tau)} = 2\int_0^\infty e^{-s\varrho(\mathrm{x}, \tau)} \partial_\mathrm{x} \varrho(\mathrm{x}, \tau) e^{-s\varrho(\mathrm{x}, \tau)} \mathrm{d}s$ is involved. Thus it is difficult to capture how interaction with an environment affects the QFI and the precision. To partially alleviate this issue, here we follow an alternative approach working with the vectorized state $\widetilde{\varrho}$ instead, which enables a bound somewhat akin to Eq. (4)—with H replaced with $\mathcal{L}$. Although an obvious expense to be paid in going from ϱ to $\widetilde{\varrho}$ is the (artificial) nonlinearity of relations in terms of ϱ, we show that this formalism still retains its relative utility in providing useful bounds. Let us briefly remark that the mentioned nonlinearity problem could be lifted if we instead worked with the purification w of $\varrho = ww^\dagger$, but this does not really offer an advantage over the original picture in leading to a straightforward uncertainly-like relation.

Now from the symmetric logarithmic derivative $L_{\widetilde{\varrho}} = 2\partial_\mathrm{x} \widetilde{\varrho}$, one can define an associated QFI $\widetilde{\mathcal{F}}^{(\mathrm{Q})}$ by replacing $(\varrho, L_\varrho) \to (\widetilde{\varrho}, L_{\widetilde{\varrho}})$ in $\mathcal{F}^{(\mathrm{Q})}$. After some straightforward algebra, using the dynamical equation Eq. (1), and assuming a linear x-dependence as $\widetilde{\mathcal{L}}(\mathrm{x}) \equiv \mathrm{x}\widetilde{\mathsf{L}}$, it can be seen that

$$\widetilde{\mathcal{F}}^{(\mathrm{Q})} = 4\tau^2 \mathrm{Cov}_{\widetilde{\varrho}}(\widetilde{\mathsf{L}}^\dagger, \widetilde{\mathsf{L}}). \quad (5)$$

This relation is analogous to Eq. (3), where instead of the Hamiltonian we have the Lindbladian generator of the open dynamics.

The QFI $\widetilde{\mathcal{F}}^{(\mathrm{Q})}$ has a natural interpretation. Recall that $\mathcal{F}^{(\mathrm{Q})}$ indeed emerges from the optimization of the Fisher information over all possible quantum measurements on the system $\varrho \in \mathcal{S}(\mathcal{H})$ [21]. Similarly then, $\widetilde{\mathcal{F}}^{(\mathrm{Q})}$ is obtained if any quantum measurement on the 'system' $\widetilde{\varrho} \in \mathcal{S}(\mathcal{H}^{\otimes 2})$ is allowed. Note, however, that a natural extension of the measurements in $\mathcal{H}$ to $\mathcal{H}^{\otimes 2}$ does not necessarily translate into most general measurements there. For example, a complete set of measurement $\{\Pi_i\}$ (with the properties $\Pi_i \geqslant 0$ and $\sum_i \Pi_i = \mathbb{1}_\mathcal{H}$), when extended simply as $\widetilde{\Pi}_i = |\Pi_i\rangle\rangle\langle\langle\Pi_i|$, do not constitute a complete set in the sense that $\sum_i \widehat{\Pi}_i \neq \mathbb{1}_{\mathcal{H}^{\otimes 2}}$ in general.

Let us see how $\widetilde{\mathcal{F}}^{(\mathrm{Q})}$ compares with $\mathcal{F}^{(\mathrm{Q})}$. First we remark that, from vectorizing the very definition of the symmetric logarithmic derivative, we have $L_{\widetilde{\varrho}} = L_\varrho \otimes \mathbb{1} + \mathbb{1} \otimes L_\varrho^T - \partial_\mathrm{x} \ln \mathrm{Tr}[\varrho^2]$. This in turn yields the following expression:

$$\widetilde{\mathcal{F}}^{(\mathrm{Q})} = \frac{2}{\mathrm{Tr}[\varrho^2]}\Big(\mathrm{Tr}[\varrho L_\varrho \varrho L_\varrho] + \mathrm{Tr}[\varrho^2 L_\varrho^2] - 2\frac{(\mathrm{Tr}[\varrho^2 L_\varrho])^2}{\mathrm{Tr}[\varrho^2]}\Big). \quad (6)$$

This form is not directly related to $\mathcal{F}^{(\mathrm{Q})}$. However, using the relation $\mathrm{Tr}[XY] \geqslant \lambda_{\min}(X)\mathrm{Tr}[Y]$ (valid for any pair of positive matrices X and Y; here $\lambda_{\min}(X)$ denotes the minimum

eigenvalue of X) and $\mathrm{Tr}[\varrho^2] \leqslant 1$, we obtain

$$\widetilde{\mathcal{F}}^{(\mathrm{Q})} > 4\lambda_{\min}(\varrho)\mathcal{F}^{(\mathrm{Q})} - F(\varrho), \tag{7}$$

where $F(\varrho)$ represents the last term in Eq. (6). This bound would be trivial where during the evolution the state of the system becomes pure. An example of such cases is when the dynamics is unitary and the initial state is pure ($|\Psi(\mathrm{x},\tau)\rangle = U(\mathrm{x},\tau)|\Psi(0)\rangle$, with unitary U). For such cases a significant simplification occurs due to $\langle\Psi(\mathrm{x},\tau)|L_\Psi|\Psi(\mathrm{x},\tau)\rangle = 0$, whence Eq. (6) reduces to $\widetilde{\mathcal{F}}^{(\mathrm{Q})} = 2\mathcal{F}^{(\mathrm{Q})}$. Therefore, in general we have

$$1/\mathcal{F}^{(\mathrm{Q})} > k/(\widetilde{\mathcal{F}}^{(\mathrm{Q})} + F), \tag{8}$$

where for nonunitary evolutions in which the state of the system never becomes pure we have $k = 4\lambda_{\min}(\varrho)$, and for unitary evolutions with a pure initial state we have $k = 2$, $F = 0$, and the inequality is replaced with equality. In addition, for unitary evolutions with a mixed initial state $\varrho(0)$, we have

$$\widetilde{\mathcal{F}}^{(\mathrm{Q})}(\mathrm{x},\tau;N) = \frac{4\tau^2}{\mathrm{Tr}[\varrho^2(0)]}\mathrm{Tr}[\varrho^2(\mathrm{x},\tau)\mathsf{H}^2 - (\varrho(\mathrm{x},\tau)\mathsf{H})^2]. \tag{9}$$

This result provides another framework for open-system estimation, alternative to the approach of Ref. [9]. A natural advantage is that here the QFI is more directly related to the generator of the dynamics, and is straightforward to compute.

Example.—Let us assume that there are N probe particles each of which only interacts with a common bath such that the interactions induce all possible k-body terms (Fig. 1) [23] in the Lindbladian as follow:

$$\mathsf{L}[\circ] = \sum_{i_1\cdots i_k} \sigma_{i_1}\cdots\sigma_{i_k} \circ \sigma_{i_1}\cdots\sigma_{i_k} - C_{N,k}\circ, \tag{10}$$

where σ_{i_j} are all the same Pauli matrix (e.g., σ^z), subscript i is the particle index, and the factor $C_{N,k} = \binom{N}{k}$ counts the number of k-body operators.

We choose the initial state of the whole N-probe system to be the maximally entangled pure state $\varrho(0) = |\Psi\rangle\langle\Psi|$ (i.e., $|\varrho(0)\rangle\rangle = |\Psi\rangle|\Psi^*\rangle$), where $|\Psi\rangle = (|\lambda_M\rangle^{\otimes N} - |\lambda_m\rangle^{\otimes N})/\sqrt{2}$, and λ_m (λ_M) is the smallest (largest) eigenvalue of σ. For odd k, $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k} \otimes \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}(|\Psi\rangle \otimes |\Psi^*\rangle) = |\Psi^\perp\rangle|\Psi^{\perp^*}\rangle$, where $|\Psi^\perp\rangle = (|\lambda_M\rangle^{\otimes N} + |\lambda_m\rangle^{\otimes N})/\sqrt{2}$. It is straightforward to see that

$$\widetilde{\mathsf{L}}|\Psi\rangle|\Psi^*\rangle = C_{N,k}(|\Psi^\perp\rangle|\Psi^{\perp^*}\rangle - |\Psi\rangle|\Psi^*\rangle), \tag{11}$$

whence $(|\Psi^\perp\rangle|\Psi^{\perp^*}\rangle - |\Psi\rangle|\Psi^*\rangle)/\sqrt{2}$ becomes an eigenvector of $\widetilde{\mathsf{L}}$ corresponding to the eigenvalue $-2C_{N,k}$. Hence

$$\widetilde{\mathcal{F}}^{(\mathrm{Q})}(\mathrm{x},\tau) = \frac{16\tau^2 C_{N,k}^2 e^{-4C_{N,k}\tau\mathrm{x}}}{[e^{-4C_{N,k}\tau\mathrm{x}} - 4e^{-2C_{N,k}\tau\mathrm{x}} + 5]^2}. \tag{12}$$

An immediate implication of this relation is that for small values of the x parameter, noting $C_{N,k} = O(N^k)$ for large Ns, a polynomial precision in the estimation can be achieved.

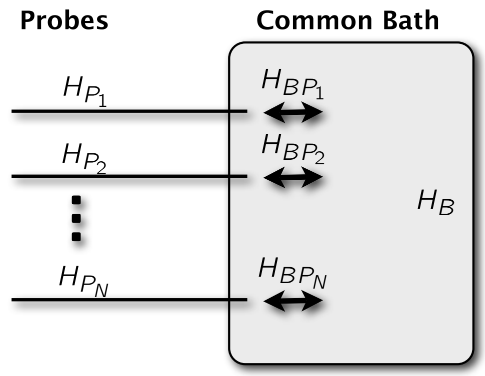

FIG. 1. N probes, initially well isolated from each other, all interact with a common bath through two-body interactions H_{BP_i}. Here H_{P_i} and H_B are the free Hamiltonians of probe i and the bath, respectively. These two-body interactions may induce a manybody quantum correlation among the probes [23].

Summary.—Here we have outlined a fairly general formalism for open quantum system metrology. In this (re)formulation of open quantum system metrology precision of estimation is more directly related to the underlying dynamics, in some sense similar to the closed system formulation. The core of this formalism is the derivation of a quantum Cramér-Rao bound for open system dynamics generated through dynamical map with the semigroup property. This setting then was illustrated with an example. This example has implied that it may be possible to exploit induced correlations of probe quantum systems through a common environment in order to achieve a relatively higher precision for an estimation task.

Our formalism may introduce methods for utilizing some of the resources offered in open quantum dynamics, such as induced manybody correlations and memory, to hopefully enhance a quantum estimation task in the presence of 'noise'. This in turn can open up possibilities for applications in , e.g., quantum sensing [17, 24] and quantum control of optomechanical devices for advanced technologies [25].

Acknowledgments.— Illuminating discussions with G. Adesso, F. Benatti, V. Karimipour, M. Mohseni, and S. Pascazio are gratefully acknowledged.

[1] P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, New York, 2003).
[2] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).
[3] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[4] P. Cappellaro, J. Emerson, N. Boulant, C. Ramanathan, S. Lloyd, and D. G. Cory, Phys. Rev. Lett. **94**, 020502 (2005);

V. Giovannetti, S. Lloyd, and L. Maccone, *ibid.*, **96**, 010401 (2006).

[5] S. Boixo, S. T. Flammia, C. M. Caves, and JM Geremia, Phys. Rev. Lett. **98**, 090401 (2007); M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell, and M. W. Mitchell, Nature **471**, 486 (2011).

[6] S. M. Roy and S. L. Braunstein, Phys. Rev. Lett. **100**, 220501 (2008).

[7] A. Rivas and A. Luis, Phys. Rev. Lett. **105**, 010403 (2010).

[8] M. Zwierz, C. A. Pérez-Delgado, and P. Kok, Phys. Rev. Lett. **105**, 180402 (2010).

[9] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nature Phys. **7**, 406 (2010).

[10] Y. Watanabe, T. Sagawa, and M. Ueda, Phys. Rev. Lett. **104**, 020401 (2010).

[11] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Application* (Springer-Verlag, Berlin, Heidelberg, 1987).

[12] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002); A. Rivas and S. F. Huelga, *Open Quantum Systems: An Introduction* (Springer, Heidelberg, 2012).

[13] X.-M. Lu, X. Wang, and C. P. Sun, Phys. Rev. A **82**, 042103 (2010).

[14] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009); A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **105**, 050403 (2010); S. Alipour, A. Mani, and A. T. Rezakhani, Phys. Rev. A **85**, 052108 (2012); K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, arXiv:1112.6238.

[15] C. H. Fleming and B. H. Hu, Ann. Phys. **327**, 1238 (2012).

[16] A. Pernice, J. Helm, and W. T. Strunz, J. Phys. B: At. Mol. Opt. Phys. **45**, 154005 (2012).

[17] V. Giovannetti, S. Lloyd, and L. Maccone, Nature Photon. **5**, 222 (2011).

[18] R. Demkowicz-Dobrzański, J. Kołodynski, and M. Guţă, Nature Commun. **3**, 1063 (2012); G. Adesso, F. DellAnno, S. De Siena, F. Illuminati, and L. A. M. Souza, Phys. Rev. A **79**, 040305(R) (2009); A. Monras and F. Illuminati, *ibid.* **83**, 012315 (2011).

[19] A. W. Chin, S. F. Huelga, and M. B. Plenio, arXiv:1103.1219.

[20] Vectorization of an operator $A = \sum_{ii'} \langle u_i|A|u_{i'}\rangle |u_i\rangle\langle u_{i'}|$ (represented in an orthonormal basis $\{|u_i\rangle\}$) is defined through $|A\rangle\rangle = \sum_{ii'} \langle u_i|A|u_{i'}\rangle |u_i\rangle|u^*_{i'}\rangle$, where $|u^*_i\rangle$ is the complex conjugate of $|u_i\rangle$ represented in the computational basis. Two related identities we use most here are $\langle\langle A|B\rangle\rangle = \mathrm{Tr}[A^\dagger B]$ and $|AXB\rangle\rangle = (A \otimes B^T)|X\rangle\rangle$, where T denotes transposition in the computational basis.

[21] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994); S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. (N.Y.) **247**, 135 (1996); M. G. A. Paris, Intl. J. Quantum Inf. **7** (Suppl.), 125 (2009); D. Braun, Eur. Phys. J. D **59**, 521 (2010).

[22] M. Hayashi, *Quantum Information: An Introduction* (Springer-Verlag, Berlin, Heidelberg, 2006).

[23] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002).

[24] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **108**, 260405 (2012).

[25] J. Abadie *et al.* (The LIGO Scientific Collaboration), Nature Phys. **7**, 962 (2011).